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Piecewise Smooth Solutions of Some Difference-Differential Equations

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1. INTRODUCTION

In this paper we consider certain difference-differential equations and look for solutions whose restriction to a given interval is a polynomial of given degree. We show that by imposing certain continuity requirements, we obtain a unique solution which coincides with a polynomial of given degree on this interval. We then show that as the degree of this polynomial tends to infinity, this solution converges to an entire function which is a solution of the given equation.

We first illustrate this procedure with a known example [5]. (For an extension of this see [4]). Let t be a complex number $t \neq 0$ or 1, arg $t \neq \pi$. Consider the functional equation

$$f(x+1) = tf(x), \qquad x \in R, f(0) = 1. \tag{1.1}$$

For $n = 1, 2, \ldots$, it is shown in [5; see also 6], that there is a unique solution S_n of (1.1) which satisfies the following two conditions:

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$$S_n \text{ coincides on } (0, 1) \text{ with a polynomial of degree } \leqslant n, \tag{1.2}$$

$$S_n \in C^{n-1} \text{ in a neighborhood of } 0. \tag{1.3}$$

We note that by (1.1), condition (1.3) implies that $S_n \in C^{n-1}(R)$. This function S_n is known as the exponential Euler spline. It is shown in [5] that for all $x \in \mathbf{R}$,

$$\lim_{n \to \infty} S_n(x) = t^x.$$

Since the function t^x is entire and satisfies (1.1), we see that the imposition of C^{n-1} continuity forces the spline solution to converge to an entire function as $n \to \infty$.

2. THE EQUATION $f(x+1) - f(x) = h(x)$

Suppose h is an entire function of exponential type $A < 2\pi$, i.e., A is the infimum of all numbers γ such that

$$h(x) = O(e^{\gamma |x|}). \tag{2.1}$$

or equivalently,

$$|h^{(\nu)}(0)| \leqslant C\gamma^\nu \qquad (\nu = 0, 1, 2,...) \text{ for some constant } C. \tag{2.2}$$

We shall apply the procedure described in Section 1 to the equation

$$f(x+1) - f(x) = h(x), \qquad x \in R, f(0) = 0. \tag{2.3}$$

The following result concerning entire solutions to (2.3) is close to that of Whittaker [7, Theorem 3, p. 22].

LEMMA 1. *There is precisely one solution of* (2.3) *which is an entire function of exponential type* $<2\pi$. *It has exponential type* A *and is given by*

$$f(x) = \sum_{\nu=0}^{\infty} \frac{h^{(\nu)}(0)}{(\nu+1)!} (B_{\nu+1}(x) - B_{\nu+1}), \tag{2.4}$$

where $B_\nu(x)$ *is the Bernoulli polynomial of degree* ν *and* $B_\nu = B_\nu(0)$ *is the corresponding Bernoulli number.*

Proof. Choose A_1 with $A < A_1 < 2\pi$. By Hadamard's theorem applied to the generating function of Bernoulli numbers, we find that there is a constant C such that

$$\frac{|B_\nu|}{\nu!} \leqslant \frac{C}{A_1^\nu} \qquad (\nu = 1, 2,...). \tag{2.5}$$

Since

$$\left|\frac{B_j(x)-B_j}{j!}\right| = \frac{1}{j!}\left|\sum_{i=1}^{j}\binom{j}{i} B_{j-i}x^i\right|$$
$$\leqslant \sum_{i=1}^{j}\frac{1}{i!}\left|\frac{B_{j-i}}{(j-i)!}\right| |x|^i,$$

it follows, on using (2.5), that

$$\left|\frac{B_j(x)-B_j}{j!}\right| \leqslant \frac{C}{A_1^j} e^{A_1|x|}.$$

Choosing A_2 such that $A<A_2<A_1<2\pi$, we conclude from (2.2) that

$$|h^{(j)}(0)| \leqslant C_1 A_2^j \qquad (j=0, 1,...). \tag{2.6}$$

From (2.6) and (2.4), we see that for all x in R,

$$|f(x)| = \left|\sum_{j=1}^{\infty}\frac{h^{(j-1)}(0)}{j!}(B_j(x)-B_j)\right|$$
$$\leqslant \sum_{j=1}^{\infty} C_1 A_2^{j-1}\frac{C}{A_1^j} e^{A_1|x|}$$
$$= C_3 e^{A_1|x|}\sum_{j=1}^{\infty}\left(\frac{A_2}{A_1}\right)^j \qquad (C_3 \text{ some constant}).$$

Since A_1 can be arbitrarily close to A, we have shown that the type of $f(x)$ is $\leqslant A$. If f had type $<A$, then by (2.3), $h(x)$ would also have exponential type $<A$. Since this is not the case, f must have exponential type A. Next recalling that

$$B_{j+1}(x+1)-B_{j+1}(x)=(j+1)\, x^j, \tag{2.7}$$

we see from (2.4) that for all $x \in R$, we have

$$f(x+1)-f(x)=\sum_{j=0}^{\infty}\frac{h^{(j)}(0)}{(j+1)!}\{B_{j+1}(x+1)-B_{j+1}(x)\}$$
$$=\sum_{j=0}^{\infty}\frac{h^{(j)}(0)}{j!}x^j = h(x).$$

Since clearly $f(0)=0$, f is a solution of (2.3). It remains to show that this is the only solution of (2.3) which has exponential type $<2\pi$. This follows from a general result that the difference of two solutions of (2.3) which are of exponential type $<2\pi$ must vanish identically. (See [1, Theorem 6.10.1, P. 109]).

THEOREM 1. *For* $n=1, 2, 3,\ldots,$ *there is a unique solution* $S_n(x)$ *of* (2.3) *which satisfies* (1.2) *and* (1.3). *Moreover* $S_n(x) \in C^{n-1}(R)$ *and converges to the function defined by* (2.4) *uniformly on* R *as* $n \to \infty$.

Proof. Let $S_n(x)$ satisfying (1.2) and (1.3) be a solution of (2.3). Suppose $S_n(x)$ coincides on $[0, 1)$ with a polynomial $P_n(x)$. Then by (2.3) we see that for $-1 \leqslant x < 0$,

$$S_n(x) \equiv P_n(x+1) - h(x).$$

Since from (1.3), $S_n(x) \in C^{n-1}$ in a neighborhood of $x=0$, we get

$$P_n^{(\nu)}(1) - h^{(\nu)}(0) = P_n^{(\nu)}(0), \qquad \nu = 0, 1,\ldots, n-1. \tag{2.8}$$

From (2.3) we also have

$$P_n(0) = 0. \tag{2.9}$$

Clearly (2.8) and (2.9) form a nonsingular system of equations to determine $P_n(x)$. Thus (1.2) and (1.3) determine a unique solution of (2.3). Clearly (2.8) and (2.3) imply that $S_n(x) \in C^{n-1}(R)$.

We now consider the polynomial

$$Q_n(x) = \sum_{j=0}^{n-1} \frac{h^{(j)}(0)}{(j+1)!} (B_{j+1}(x) - B_{j+1}) \tag{2.10}$$

which, because of (2.7), satisfies

$$Q_n(x+1) - Q_n(x) = \sum_{j=0}^{n-1} \frac{h^{(j)}(0)}{j!} x^j, \qquad x \in R. \tag{2.11}$$

Hence

$$Q_n^{(\nu)}(1) - Q_n^{(\nu)}(0) = h^{(\nu)}(0), \qquad \nu = 0, 1,\ldots, n-1. \tag{2.12}$$

Since from (2.10), $Q_n(0) = 0$, a comparison of (2.8) and (2.9) shows that $Q_n(x) = P_n(x)$.

Comparing (2.10) and (2.4) shows that $Q_n(x)$ converges uniformly in $[0, 1]$ to f, i.e., $S_n(x)$ converges uniformly to $f(x)$ on $[0, 1]$. Since $f(x)$ and $S_n(x)$ both satisfy (2.3), we get

$$f(x+1) - S_n(x+1) = f(x) - S_n(x), \qquad x \in R$$

and so $S_n(x)$ converges uniformly to $f(x)$ on R. ∎

3. A General Linear Difference-Differential Equation

We shall denote by $\mathscr{F}$ the space of entire functions f such that $\{f^{(j)}(0)\}_{j=0}^{\infty} \in l_2$. If f is an entire function of exponential type <1, then $f \in \mathscr{F}$. If $f \in \mathscr{F}$, then f is of exponential type $\leqslant 1$. For any given function $h(x) \in \mathscr{F}$ and given complex numbers $\lambda_1, ..., \lambda_p$, $\mu_1, ..., \mu_q$ with $\lambda_p \mu_q \neq 0$, we shall apply our procedure to the equation

$$\sum_{i=0}^{p} \mu_i f^{(i)}(x+1) = \sum_{i=0}^{q} \lambda_i f^{(i)}(x) + h(x), \qquad x \in R, f(0) = 1. \tag{3.1}$$

The key tool will be the following result which is taken from Theorem 7.1 (Sect. 1) and Theorem 2.1 of Chapter 3 of [2].

Theorem A (Gokhberg and Feldman[2]). *Let $a(z)$ be an arbitrary function continuous on the unit circle and a_j $(j = 0, \pm 1, ...)$ its Fourier coefficients. Let the operator $A: l_2 \to l_2$ be defined by*

$$(Ab)_j = \sum_{k=1}^{\infty} a_{j-k} b_k \qquad j = 1, 2, ...; b = \{b_j\}_1^{\infty} \in l_2.$$

Then A is invertible if and only if

$$a(z) \neq 0 \qquad \text{for} \quad |z| = 1, \tag{3.2}$$

$$\operatorname{ind} a := \frac{1}{2\pi} \arg a(e^{i\theta})]_0^{2\pi} = 0. \tag{3.3}$$

Moreover, if (3.2) and (3.3) are satisfied, then the following holds: The system of equations

$$\sum_{k=1}^{n} a_{j-k} \xi_k^{(n)} = \eta_j, \qquad j = 1, ..., n \tag{3.4}$$

is nonsingular for all large enough n. If $\{\eta_j\}_1^{\infty} \in l_2$, $\{\xi_k^{(n)}\}_{k=1}^{n}$ denotes the solution of (3.4) and if $\xi^{(n)} := \{\xi_1^{(n)}, ..., \xi_n^{(n)}, 0...\}$, then $\xi^{(n)}$ converges in l_2 to the unique solution $\xi := \{\xi_j\}_{j=1}^{\infty} \in l_2$ of the infinite system of equations

$$\sum_{k=1}^{\infty} a_{j-k} \xi_k = \eta_j \qquad (j = 1, 2, ...). \tag{3.5}$$

We shall now prove

Theorem 2. *Suppose the function*

$$a(z) := e^z \sum_{i=0}^{p} \mu_i z^{i-1} - \sum_{i=0}^{q} \lambda_i z^{i-1} \tag{3.6}$$

satisfies conditions (3.2) *and* (3.3) *of Theorem A. Then for large enough n, there is a unique solution* $S_n(x)$ *of* (3.1) *which satisfies the conditions* (1.2) *and also*

$$S_n(x) \in C^{n-1+q} \quad \text{in a neighbourhood of } 0, \tag{3.6a}$$

$$S_n(x) \in C^{p-1} \quad \text{in } (0, \infty), \tag{3.7}$$

$$S_n(x) \in C^{q-1} \quad \text{in } (-\infty, 0). \tag{3.8}$$

Moreover, there is a unique solution $f(x)$ *of* (3.1) *in* $\mathfrak{F}$ *and for all* $x \in R$, $S_n(x) \to f(x)$ *as* $n \to \infty$.

Proof. Suppose $S_n(x)$ coincides on $(0, 1)$ with a polynomial $P_n(x)$, where

$$P_n(x) = 1 + b_1^{(n)} x + \frac{b_2^{(n)}}{2!} x^2 + \cdots + \frac{b_n^{(n)} x^n}{n!}. \tag{3.9}$$

Suppose moreover that $S_n(x)$ also satisfies (1.3), (3.7), and (3.8) and is a solution of (3.1). Then by (3.1) we see that for $-1 \leqslant x \leqslant 0$,

$$\sum_{i=0}^{q} \lambda_i S_n^{(i)}(x) = \sum_{i=0}^{p} \mu_i P_n^{(i)}(x+1) - h(x).$$

Differentiating v times and letting $x \to 0^-$, gives

$$\sum_{i=0}^{q} \lambda_i S_n^{(i+v)}(0^-) = \sum_{i=1}^{p} \mu_i P_n^{(i+v)}(1) - h^{(v)}(0), \qquad v = 0, 1, \ldots, n-1.$$

Since $S_n(x) \in C^{n-1+q}$ in a neighborhood of 0, this is equivalent to

$$\sum_{i=0}^{p} \mu_i P_n^{(i+v)}(1) - h^{(v)}(0) = \sum_{i=0}^{q} \lambda_i P_n^{(i+v)}(0) \qquad (v = 0, \ldots, n-1). \tag{3.10}$$

Substituting (3.9) into (3.10) and adopting the usual convention that the reciprocal of the factorial of a negative integer is zero, we obtain the following system of n equations ($v = 0, 1, \ldots, n-1$):

$$\sum_{k=1}^{n} b_k^{(n)} \sum_{i=0}^{p} \frac{\mu_i}{(k-i-v)!} - \sum_{i=0}^{q} \lambda_i b_{i+v}^{(n)} = h^{(v)}(0) + \alpha_v, \tag{3.11}$$

where

$$\alpha_v = \lambda_0 - \mu_0, \qquad v = 0,$$
$$= 0, \qquad v > 0.$$

The system (3.11) can be written as

$$\sum_{k=1}^{n} a_{j-k} b_k^{(n)} = h^{(j-1)}(0) + \alpha_{j-1}, \qquad j = 1,..., n, \tag{3.12}$$

where

$$a_l = \sum_{i=0}^{p} \frac{\mu_i}{(1-l-i)!} - \sum_{i=0}^{q} \lambda_i \delta_{i,1-l}, \qquad l \in \mathbf{Z}. \tag{3.13}$$

Comparing with (3.4) and applying the first part of Theorem A shows that the system (3.12) is nonsingular for large enough n provided conditions (3.2) and (3.3) of Theorem A are satisfied. Thus there is a unique polynomial $P_n(x)$ satisfying (3.10) and so conditions (1.2) and (3.6a) determine the solution $S_n(x)$ uniquely on $[0, 1)$. Applying (3.1) and condition (3.7), we can extend $S_n(x)$ uniquely to a solution on $(0, \infty)$. Similarly applying (3.1) and condition (3.8), $S_n(x)$ can be extended to a solution on $(-\infty, 0)$.

By an argument similar to the above we see that an entire function

$$f(x) = 1 + \sum_{k=1}^{\infty} b_k x^k/k! \tag{3.14}$$

is a solution of (3.1) if and only if

$$\sum_{k=1}^{\infty} a_{j-k}\, b_k = h^{(j-1)}(0) + \alpha_{j-1}, \qquad (j = 1, 2,...), \tag{3.15}$$

where a_l is given by (3.13).

By Theorem A we know that there is a unique solution $\mathbf{b} = \{b_k\}_1^\infty$ of (3.15) with $\{b_k\}_1^\infty \in l_2$, and thus there is a unique solution $f \in \mathscr{F}$ of (3.1). Again, by Theorem A we know that $\mathbf{b}^{(n)} := \{b_1^{(n)},..., b_n^{(n)}, 0, 0,...\}$ converges in l_2 to $\mathbf{b}$. Now for $x \in R$,

$$\begin{aligned} |f(x) - P_n(x)| &= \left| \sum_{k=1}^{\infty} (b_k - b_k^{(n)}) \frac{x^k}{k!} \right| \\ &\leqslant \|\mathbf{b} - \mathbf{b}^{(n)}\|_2 \left\| \left\{ \frac{x^k}{k!} \right\}_1^\infty \right\|_2 \end{aligned}$$

and so $f(x) - P_n(x) \to 0$ as $n \to \infty$.

We have thus shown that $S_n(x) \to f(x)$ as $n \to \infty$ for x in $[0, 1)$ and applying (3.1) shows that $S_n(x) \to f(x)$ as $n \to \infty$ for all x in R. ∎

Remark. We note that Eqs. (1.1) and (2.3) are special cases of (3.1). However the results mentioned in Section 1 cannot be deduced from

Theorem 2 because in this case the function $a(z)=(e^z-t)/z$ does not satisfy conditions (3.2) and (3.3) of Theorem A for all values of t. Similarly the results of Section 2 cannot be deduced from Theorem 2 because the conditions on $h(x)$ are weaker.

4. An Example

We now consider the particular equation

$$f(x)=f'(x+1), \qquad x\in R, f(0)=1 \tag{4.1}$$

as an illustration of Theorem 2 because the solution in this case has an interesting form. The function $a(z)$ given by (3.6) is in this case

$$a(z)=e^z-z^{-1}. \tag{4.2}$$

Now Im $a(e^{i\theta})=e^{\cos\theta}\sin(\sin\theta)+\sin\theta>0$ for $0<\theta<\pi$. Since $a(e^{i\theta})$ is real and positive for $\theta=0$ and π, we see that $a(e^{i\theta})\neq 0$ for $0<\theta<\pi$ and arg $a(e^{i\theta})]_0^\pi=0$. Similarly, $a(e^{i\theta})\neq 0$ for $\pi\leqslant\theta\leqslant 2\pi$ and arg $a(e^{i\theta})]_\pi^{2\pi}=0$. Thus $a(z)$ satisfies conditions (3.2) and (3.3) of Theorem A and we can apply Theorem 2. It is easily seen that a solution f of (4.1) which lies in $\mathscr{F}$ is$f(x)=e^{\lambda x}$, where $1=\lambda e^\lambda$ (In fact $\lambda\approx 0.5671432904$.). Thus Theorem 2 tells us that

$$\lim_{n\to\infty} S_n(x)=e^{\lambda x}, \qquad x\in R.$$

We now find an explicit expression for the spline $S_n(x)$. In this case, Eq. (3.3) becomes

$$P_n^{(\nu+1)}(1)=P_n^{(\nu)}(0) \qquad (\nu=0, 1,..., n-1). \tag{4.3}$$

It is easily checked that for $n\geqslant 0$, this is satisfied by

$$P_n(x)=\sum_{j=0}^{n}\frac{(x+n-j)^j}{j!}. \tag{4.4}$$

Since we must have $S_n(0)=1$, (4.4) yields

$$S_n(x)=\gamma_n P_n(x), \qquad 0\leqslant x\leqslant 1, \tag{4.5}$$

where

$$\gamma_n^{-1}=\sum_{j=0}^{n}\frac{(n-j)^j}{j!}. \tag{4.6}$$

We now extend $S_n(x)$ uniquely to $[-1, 0)$ by applying condition (4.1). Thus

$$S_n(x) = S_n'(x+1) = \gamma_n P_n'(x+1) = \gamma_n P_{n-1}(x+1), \qquad -1 \leqslant x < 0.$$

Similarly we extend $S_n(x)$ uniquely to $[1, 2)$ by applying (4.1) and condition (3.7) of Theorem 2:

$$\begin{aligned} S_n'(x) &= S_n(x-1) = \gamma_n P_n(x-1), \qquad 1 \leqslant x < 2 \\ S_n(1) &= \gamma_n P_n(1) = \gamma_n P_{n+1}'(1) = \gamma_n P_{n+1}(0). \end{aligned} \tag{4.7}$$

We easily see that (4.7) is satisfied by

$$S_n(x) = \gamma_n P_{n+1}(x-1), \qquad 1 \leqslant x < 2.$$

In this way we see by successive extensions to the right and to the left that

$$S_n(x) = \gamma_n P_{n+\nu}(x-\nu), \qquad \nu \leqslant x < \nu+1,\ \nu \in Z, \tag{4.8}$$

where $P_\mu \equiv 0$ for $\mu < 0$. A more compact and neat expression for $S_n(x)$ is given by

$$S_n(x) = \gamma_n \sum_{\nu=0}^{\infty} \frac{(x+n-\nu)_+^\nu}{\nu!} = \gamma_n S_0(x+n), \tag{4.9}$$

where $S_0(x) = \sum_{\nu=0}^{\infty} (x-\nu)_+^\nu / \nu!$.

5. A Conjecture

We close this note by mentioning the nonlinear equation

$$f(x) = x f(x+1), \qquad x \in R,\ f(1) = 1, \tag{5.1}$$

which is satisfied by the entire function $f(x) = 1/\Gamma(x)$. It is shown in [3] that for $n = 1, 2, \ldots$, there is a unique solution $S_n(x)$ of (5.1) satisfying (1.2) and (1.3). We offer the following

Conjecture. For all $x \in R$, $S_n(x) \to 1/\Gamma(x)$ *as* $n \to \infty$. We remark that writing

$$S_n(x) = x + \frac{b_2^{(n)}}{2!} x^2 + \cdots + \frac{b_n^{(n)} x^n}{n!}, \qquad 0 \leqslant x \leqslant 1,$$

the coefficients $b_2^{(n)},\ldots, b_n^{(n)}$ are the unique solution of the system of equations

$$\sum_{j=2}^{n} \left\{\frac{1}{(j-k)!} - \frac{1}{k+1}\,\delta_{j-k,1}\right\} b_j^{(n)} = -\delta_{k,1}, \qquad k=0, 1,\ldots, n-2.$$

which is very similar to the system of equations (3.4) in Theorem A. The validity of this conjecture would imply that as $n \to \infty$, $\frac{1}{2} b_2^{(n)} \to \gamma$, the Euler constant, a fact which is strongly supported by numerical evidence. For further details on this conjecture see [3].

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